A limit of toric symplectic forms that has no periodic Hamiltonians

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Abstract

We calculate the Riemann-Roch number of some of the pentagon spaces defined in [Kl, KM, HK1]. Using this, we show that while the regular pentagon space is diffeomorphic to a toric variety, even symplectomorphic to one under arbitrarily small perturbations of its symplectic structure, it does not admit a symplectic circle action. In particular, within the cohomology classes of symplectic structures, the subset admitting a circle action is not closed.

1 The result

In [Kl, HK1, KM] are introduced *polygon spaces*, which are a family of symplectic manifolds often possessing many interesting circle actions. In the case of pentagons with distinct edge-lengths, they are actually (symplectomorphic to) toric varieties. So it is rather a surprise that the regular pentagon space, although *diffeomorphic* to the nearby nearly-regular pentagon spaces, admits no symplectic circle action at all.

In what follows we calculate the Riemann-Roch number of the regular pentagon space, 6, and its Euler characteristic, 7. This will already be enough to show it isn't a toric variety. To prove the stronger statement we invoke results of Karshon [Ka] on arbitrary symplectic 4-manifolds with circle actions.

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2 Nearly-regular pentagon spaces and toric structures

We remind the reader of the definition of pentagon spaces and why they are frequently symplectomorphic to toric varieties [HK1, KM]. In what follows our cohomology groups will always be with real coefficients.

2.1 Pentagon spaces.

Given a list $(a_1, \ldots, a_5)$ of positive real numbers, define the *pentagon space* $\mathrm{Pol}\,(a_1, \ldots, a_5)$ as the space of 5-step closed piecewise-linear paths in $\mathbf{R}^3$ starting at the origin, with step-lengths $a_1, \ldots, a_5$, considered up to $SO(3)$-rotation. As such, it is a subquotient of the *path space* $\prod_{i=1}^5 S^2_{a_i}$ of 5-step paths in $\mathbf{R}^3$ starting at the origin, not necessarily closing up nor quotiented by $SO(3)$.

Regarding $S^2_{a_i}$ as a symplectic manifold with area a_i, this construction can be interpreted as producing the pentagon space as a symplectic quotient of the path space by $SO(3)$. Consequently, the pentagon space is also symplectic when it is smooth; this happens exactly if no two of the edges put together have the same total length as the other three put together.

2.2 Nearly-regular pentagon spaces.

Definition 2.1 *We define a* nearly-regular *pentagon space* $\mathrm{Pol}\,(a_1, \ldots, a_5)$ *to be one in which the sum of any two edge-lengths $a_i + a_j$ is* less *than the sum of the other three.*

In particular these spaces are smooth, and of course include the regular pentagon space $\mathrm{Pol}\,(1, 1, 1, 1, 1)$.

As we will see below, any nearly-regular pentagon space is diffeomorphic to $(S^2 \times S^2) \# \overline{CP^2} \# \overline{CP^2} \# \overline{CP^2}$; so when we now restrict attention to nearly-regular pentagon spaces, we can view this as studying different symplectic structures on the same manifold.

2.3 Toric structures.

Take now the generic case that $a_1 \neq a_2$, $a_4 \neq a_5$.

In [Kl, KM] are defined the *bending flows* on such a polygon space. Label the vertices A,B,C,D,E, and draw the diagonals AC and AD, breaking the pentagon into three triangles ABC, ACD, and ADE. Since these diagonals

cannot be zero length (by the genericity assumption above), each one gives a circle action: break off the triangles ABC and ADE and reattach them rotated by independent angles. In fact these circle actions on this symplectic manifold are Hamiltonian, with the Hamiltonians given by the lengths of the diagonals (d_1, d_2).
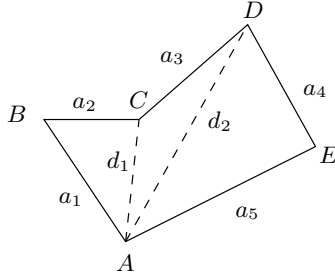
In [HK1, p. 192] we calculated the image of the map (d_1, d_2) in $\mathbf{R}^2$; in the nearly-regular case we always get a rectangle missing three corners. (See figure 1 below.) With this and Delzant's reconstruction theorem it is immediate that the space is diffeomorphic to $(S^2 \times S^2) \# \overline{CP^2} \# \overline{CP^2} \# \overline{CP^2}$. In particular the 7 vertices says that the Euler characteristic is 7.
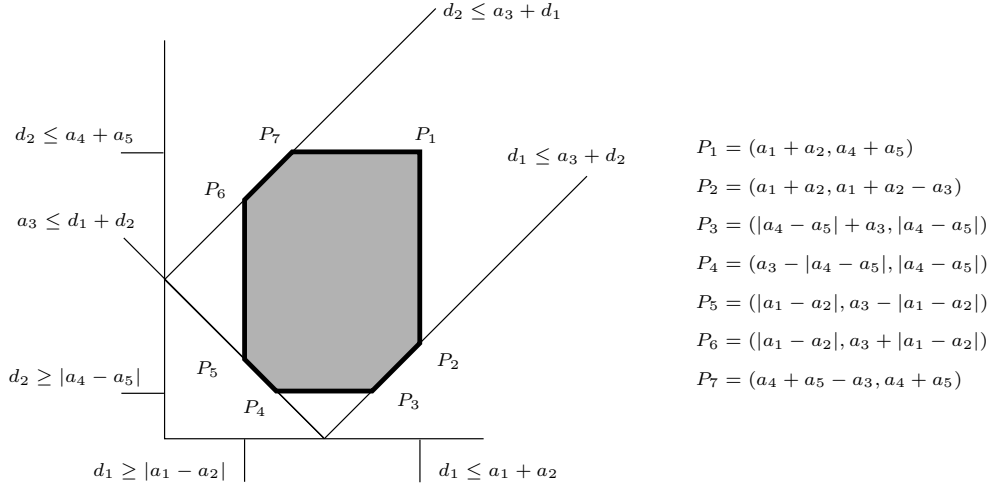

$$P_1 = (a_1 + a_2, a_4 + a_5)$$
$$P_2 = (a_1 + a_2, a_1 + a_2 - a_3)$$
$$P_3 = (|a_4 - a_5| + a_3, |a_4 - a_5|)$$
$$P_4 = (a_3 - |a_4 - a_5|, |a_4 - a_5|)$$
$$P_5 = (|a_1 - a_2|, a_3 - |a_1 - a_2|)$$
$$P_6 = (|a_1 - a_2|, a_3 + |a_1 - a_2|)$$
$$P_7 = (a_4 + a_5 - a_3, a_4 + a_5)$$

Figure 1: The moment polytope of $\mathrm{Pol}\,(a_1, \ldots, a_5)$.

Actually one can do slightly better than this abstract diffeomorphism.

3

Fix $(M,\omega) := Pol(1,1,1,1,1)$. Then for $a := (a_1,...,a_5)$ the edge-lengths of a nearly-regular pentagon space there is a canonical (up to isotopy) diffeomorphism $h : M \to Pol(a_1,...,a_5)$ ([HK2, Prop. 2.2]). Call $\omega_a \in H^2(M)$ the (well-defined) class $\omega_a := h^*\omega$.

2.4 Riemann-Roch numbers of nearly-regular pentagon spaces.

Any compact symplectic manifold (M,ω) has a canonical "Todd class" $\operatorname{Td} M$, constructed by endowing the manifold with an auxiliary almost complex structure. (In fact pentagon spaces are actually complex [Kl].) The *Riemann-Roch number* $RR(M,\omega)$ is defined as $\int_M \exp([\omega])\operatorname{Td} M$ and only depends on the cohomology class $[\omega]$ of the symplectic form.

Note that we have not assumed integrality of ω for this definition; rather, this is a polynomial function on $H^2(M)$ which merely happens to take on integral values at integral ω. In the case of pentagon spaces this exactly corresponds to the side-lengths being integral [HK2, corollary 6.6, p. 304].

Lemma 2.2 *As in the last section, fix M the regular pentagon space, but change the symplectic form to ω_a, where $\operatorname{Pol}(a_1,\ldots,a_5)$ is a toric, integral, nearly-regular pentagon space. Then the Riemann-Roch number of (M,ω_a) is*

$$\frac{1}{2}(a_1+a_2+a_3+a_4+a_5)^2 - 2(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2) + \frac{1}{2}(a_1+a_2+a_3+a_4+a_5)+1.$$

Proof. If M is a toric manifold with moment map $\mu : M \to \mathbf{t}^*$, ω integral, and some (therefore every) fixed point of M maps to a lattice point in $\mathbf{t}^*$ under μ, then the Riemann-Roch number of M is equal to the number of lattice points in the image of μ [Fu].

In our case the moment map is $\mu = (d_1, d_2)$. Assume for convenience that $a_1 > a_2$ and $a_4 > a_5$ (they may be switched if necessary without affecting the Riemann-Roch number). Then by Pick's theorem [Fu, p. 113] the number of lattice points in figure 1 is the area plus one-half the number of lattice points on the boundary plus one. The area is

$$4a_2a_5 - \frac{1}{2}(a_2+a_4+a_5-a_1-a_3)^2 - \frac{1}{2}(a_2+a_3+a_5-a_1-a_4)^2 - \frac{1}{2}(a_1+a_2+a_5-a_3-a_4)^2$$

and the boundary has

$$2(2a_2+1)+2(a_1-a_2)-1+a_3+a_4+a_5-a_1-a_2-1 = a_1+a_2+a_3+a_4+a_5$$

4

lattice points. Expanding and collecting terms we get the result (which is, as it should be, symmetric in $a_1 \ldots a_5$). ▯

Corollary 2.3 *The formula from lemma 2.2 holds for any nearly-regular pentagon space; in particular $RR(\mathrm{Pol}\,(1,1,1,1,1)) = 6$.*

Proof. Let C be any open cone in $\mathbf{R}^n$ of dimension n, i.e. an open subset closed under multiplication by positive reals. (For us, the relevant $\mathbf{R}^n$ will be $H^2(M)$, and C the space of classes of symplectic forms from nearly-regular pentagon spaces.)

We first observe that a polynomial p on $\mathbf{R}^n$ is determined by its values on $C \cap \mathbf{Z}^n$. Proof: p is dominated by its highest-order terms, p_{top}. If p_{top} vanishes on $C \cap \mathbf{Z}^n$, it vanishes on $C \cap \mathbf{Q}^n$ by homogeneity, therefore on all of C by continuity, therefore everywhere by analyticity.

Next we find a basis of $H^2(M)$ with which to apply this. Recall that we constructed the pentagon space by reduction from the 5-step path space $(S^2)^5$ by $SO(3)$. Since $H^2(BSO(3))$ is trivial (recall we're using real coefficients), by Kirwan's theorem the linear map from $H^2_{SO(3)}((S^2)^5)$ to $H^2(M)$ is surjective. By dimension count it is an isomorphism (the moment polytope lets us determine that $b_2(M)$ is also 5). So the obvious basis of $H^2((S^2)^5)$ – which is what we were using to write down the vectors $(a_1, \ldots, a_5)$ – gives us a basis for $H^2(M)$.

Since the Riemann-Roch number of (M, ω) is a polynomial function on $H^2(M)$, and has been determined on the subset of integral, positive, distinct $(a_1, \ldots, a_5)$ in lemma 2.2, the formula in lemma 2.2 in fact calculates the Riemann-Roch number for any nearly-regular pentagon space. ▯

It is worth mentioning a corollary: the symplectic volume (which is always the leading term in the Riemann-Roch number) of a nearly-regular pentagon space is $\frac{1}{2}(a_1 + a_2 + a_3 + a_4 + a_5)^2 - 2(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)$.

3 The regular pentagon space has no periodic Hamiltonians

Let $P := \mathrm{Pol}\,(1,1,1,1,1)$. So far we know its Riemann-Roch number is 6 and its Euler characteristic is 7.

Lemma 3.1 *P is not a toric manifold.*

Proof. Suppose that P admits a Hamiltonian action of a 2-torus T. Let Π be the moment polytope in $\mathrm{Lie}\,(T)^*$. As P is an integral symplectic manifold, one may translate Π so that its vertices are lattice points. Since P's Euler characteristic is 7, Π has 7 vertices, and thus at least 7 lattice points. This contradicts the fact that $RR(P) = 6$, since the Riemann-Roch number of a toric manifold is the number of lattice points in its moment polytope. ⬚

Theorem 3.2 *P admits no periodic hamiltonian.*

Proof. Suppose that $f : P \to \mathbf{R}$ is a moment map for the Hamiltonian action of a circle on P.

One may suppose that the minimum value of f is 0. As P is an integral symplectic manifold, the critical values of f are then integers (see [Ka, Lemma 2.5] and its proof). Observe that there are more than two critical values. Otherwise, P would be CP^2 or an S^2-bundle over S^2 ([Au, 2.4.2]), neither of which fit with $b_2(P) = 5$. Also, at least one of the extremal level sets (say the minimal one) is not an isolated point. Otherwise, f would have only isolated critical points and then be a toric manifold by [Ka, Theorem 5.1], which contradicts the lemma above.

We now consider the Duistermaat-Heckman function DH for f which is a piecewise affine map from $\mathbf{R}$ to $\mathbf{R}_{\geq 0}$. We shall use the following facts:

1. $DH(0)$ is the area of the sphere $f^{-1}(0)$ which is an integer since ω is integral.

2. The slope of DH at 0 is the Euler class of the normal bundle of $f^{-1}(0)$ in P ([Ka, Lemma 2.12]).

3. the slope of DH changes exactly at each critical value of f and the graph of DH is concave ([Ka, Lemmas 2.12 and 2.19].

4. the number of integral points enclosed by the graph of DH is $RR(P) = 6$.

One checks that the only possible DH function must then satisfy

$$DH(0) = 1, \quad DH(1) = 2 \ \text{ and } DH(y) = 0 \ \text{ for } y \geq 2.$$

Therefore, the maximum level $f^{-1}(2)$ would be an isolated point. But the slope of DH at an extremal isolated point should be ≤ 1 in absolute value ([Ka, Lemma 2.12]) which is not the case here. ⬚

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